

Exotic smoothness and spacetime models

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Introduction

“**Exotic**” here describes certain mathematical facts that are surprising and highly counter-intuitive. They often follow on the construction of counter examples to assumptions that seem very reasonable, especially to physicists.

Motivation for considering exotic smoothness

We begin by pointing out that progress in theoretical physics has often come as a result of questioning old assumptions, e.g.,

- *spacetime should be an absolute product, time \times space,*
- *spacetime should be geometrically flat,*
- *spacetime should have trivial topology,*

and many others. Questioning these natural assumptions obviously has led to many rich discoveries.

Topics

In this questioning spirit, it would seem to be well worthwhile to explore the recent discovery of exotic differentiable structures on topologically trivial spaces, especially \mathbb{R}^4 .

- We start with a brief discussion of what **spacetime structures** are needed to do physics on a spacetime model.
- We then explore some of the foundations of **differential topology**, which has generally been assumed to have only trivial implications for physics.
- We look at the notions associated with **relativity principles** at various levels.
- We look at some easily displayed **examples of exotica**: Weierstrass functions, complex structures on \mathbb{R}^2 , Whitehead spaces, and Milnor spheres.
- Next we mention some of the highlights in the **discovery of exotic \mathbb{R}^4 's**, which we denote by \mathbb{R}_{Θ}^4 .
- Some results of possible geometrical and **physical significance** for \mathbb{R}_{Θ}^4 's are presented.

Spacetime Structures

To do physics, we need some model of space time, including at least:

1. **Point set**
2. **Topology**
3. **Smoothness, differentiability, C^∞**
4. **Geometry, bundle structures, etc.**

Until now, this transition,

$$\textit{point set} \rightarrow \textit{topological space} \xrightarrow{??} \textit{smooth manifold} \rightarrow \textit{bundles, etc.}$$

was thought to be fairly well understood, and explored.

However, there has recently been a big surprise, the discovery of **exotic smoothness**. Now it is known that spaces with even very simple topology can support a myriad of unequivalent smoothness structures. This calls into question the triviality of the transition,

$$\textit{topological space} \xrightarrow{??} \textit{smooth manifold}.$$

So, we review the subject of **Differential Topology**, concerned with this issue.

Differential Topology = Global Calculus

In defining any point set, X , there may not be a priori any *preferred* way to associate numbers with a given point, $\mathbf{p} \in X$. For spacetime models, \mathbf{p} is an **event**. The process of assigning numbers is determined by the physical procedure associated with a **reference frame**, mathematically by a **coordinate patch**.

It is easy to get lazy and falsely secure about this matter since most spaces, X , considered in both physics and mathematics are modeled by subsets of \mathbb{R}^n , so each \mathbf{p} is “naturally” associated with an ordered set of real numbers. However, it is well known that

Remark 1 The definition of coordinates is not unique.

From this arises the following question at the heart of the **principle of general relativity**:

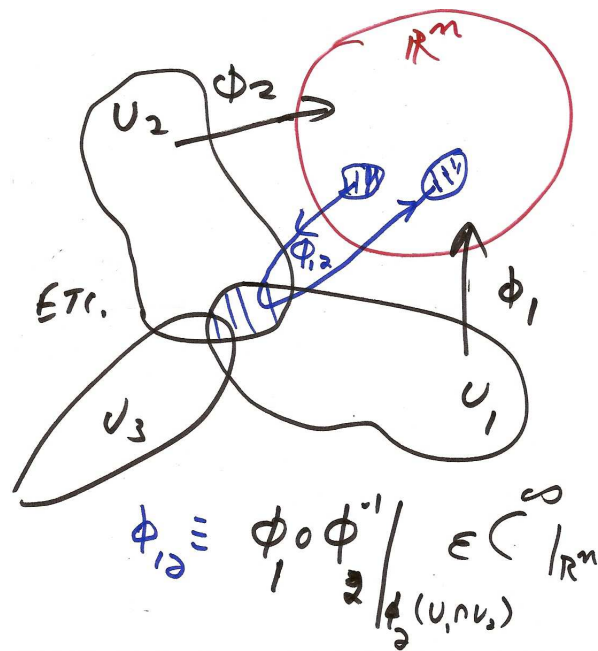
Question 1 Does re-coordination have any physical (or mathematical) consequences?

Process of assigning numbers to points in X is accomplished by an **atlas of charts**, $(U, \phi_U : U \rightarrow \mathbb{R}^n)$. To do calculus, need differential consistency in overlap, $\phi_U \cdot \phi_V^{-1} \in C^\infty$. A **differentiable structure** on X is defined by a maximal atlas, and makes X into a **differentiable**, or **smooth manifold**. The atlas enables the consistent definition of calculus over X , obviously indispensable for any physical theory.

The atlas contains coordinate transitions *within* a given X , but also allows the definition of a natural *equivalence* established by a **diffeomorphism**. This is a homeomorphism of one smooth manifold to another (or itself), $f : X \rightarrow Y$, which together with its inverse is smooth when expressed in the atlases on X and Y respectively.

Question 2 Does mathematical equivalence (homeomorphism) uniquely determine physical equivalence at the topological level?

Smooth Coordinate Presentation Problem



This figure illustrates the basic problem of smooth coordinate patching. Most of physics is done in local coordinates, which must be smoothly patched together according to the rules described by the definition of the smooth manifold. For many of our **exotic** manifold examples, this image is part of an infinite family of coordinate neighborhoods. This pattern cannot be reduced to a single patch in the case of \mathbb{R}_{\ominus}^4 , and probably not even to a finite number of patches.

Another way of looking at this is to ask if a single coordinate neighborhood can be extended indefinitely. We will discuss this later in the context of analogous questions raised by the singularities of the Schwarzschild metric expressed in its original (t, r) coordinates.

Relativity Principles

Relativity principles state that our description of reality should not depend on certain details of how we describe it, e.g., choice of reference frame, etc., but other choices are available to various formalisms. Before beginning we should recall of course that the converse of a relativity principle states that models that cannot be related by the permitted “coordinations” are indeed **physically inequivalent**.

- **Topological relativity:** Here we might assert that our description of physics should be equally valid in any (topological) coordinate system that preserves the **topology** of \mathcal{M}_{TOP} . This is invariance under **homeomorphisms**. Mathematically, this means we must allow any coordinate system which ascribes $\{y^\mu\}$ to points in such a way that the defining functions,

$$\{p^\mu\} \rightarrow \{y^\mu\} = f^\mu(p^\nu)$$

provide a **homeomorphism**, which, by definition, preserves the topology of the model. From Descartes to Newton to the early days of Einstein theory we have assumed that this choice of standard, Euclidean, topology was the appropriate one (and probably the only appropriate one) to be used for spacetime models. Of course, mathematicians have long been interested in spaces with non-euclidean models but it seems it was only when careful consideration was given to certain classes of solutions of the Einstein general relativistic gravitational equations, e.g., 3-sphere closed space cosmology, that alternatives were considered by physicists.

Even at this elementary level, however, we find that we have made a choice of structure for the point set, \mathbb{R}^4 . That is: **why choose the “standard” topology for \mathbb{R}^4 ?** In fact, there are infinitely many other notions of “nearness” or topology that could be used to define our spacetime model. For example, as a point set, \mathbb{R}^4 is **isomorphic** to \mathbb{R}^1 , so why not choose this as the basis for the “physical” topology of \mathbb{R}^4 ? This isomorphism was perhaps first made explicit by Peano and relates to set theoretic work of Cantor. One way to define the isomorphism is to interweave the digits representing each of the four numbers, mapping them uniquely into a single number.

Question 3 But, given this arbitrariness, what physical principle leads us to choose the product topology of $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ for our spacetime model of events? Why do we choose $\mathcal{M}_{TOP} = \mathbb{R}^4$ for our space time topology, rather than $\mathcal{M}_{TOP} = \mathbb{R}^1$ defined by a Cantor-Peano space filling curve? Is this a result of human intuition, or is there a physical empirical basis?

Of course, no viable theory has used this approach to topologizing the mathematical representation of the four spacetime coordinates, but it is important to realize that an *a priori* choice has been made, and that our topological relativity principle, invariance under homeomorphisms, is restricted to this original choice.

Again the converse of this relativity is that two spacetime models with **non-homeomorphic** spacetime models are truly physically inequivalent.

Once the topology has been chosen, we need an additional structure to express point particle kinematics and field theories locally as differential equations. In fact, the physics of Newtonian mechanics was instrumental in the early development of calculus. However, to do calculus we need to know which coordinates are smooth. A choice of which coordinates are to be **smooth** converts \mathcal{M}_{TOP} into \mathcal{M}_{SMOOTH} . Given this we are naturally led to another relativity principle:

- **Smooth relativity:** Here the laws of physics must be invariant under smooth transformations of coordinates. That is, if x^μ have been chosen to define smooth coordinates on \mathcal{M}_{TOP} converting it to a \mathcal{M}_{SMOOTH} then the laws of physics must be equally valid in any other coordinate system, y^μ , if the maps defining the y^μ in terms of the x^ν are **diffeomorphisms**. Actually, this can be taken as a definition of Einstein's General Principle of Relativity, but we are getting a little bit ahead of ourselves here because in the meantime flat models of space and spacetime were dominant and thought to be physically appropriate.

Again,

Remark 2 Non-diffeomorphic spacetime manifolds are physically inequivalent.

Before General Relativity, it seems to have been generally thought that the mathematics of Euclidean geometry for space and Lorentzian geometry for spacetime were the only physically appropriate choices.

- **Geometric relativity:** So, with special relativity Einstein suggested that the laws of physics required the flat spacetime Minkowski metric so that a restricted class of coordinate systems in which this metric assumed its canonical form were physically preferred. These are the inertial reference frames of Special Relativity. The generalization of this to a **Principle of General Relativity** led to arbitrary, but *smooth* coordinate systems. It is of course well understood now that in the context of spacetime this leads to “accelerated” reference frames, freely falling elevators, etc., and eventually to the suggestion that gravity could be associated with a non-trivial metric, **Einstein’s General Relativistic Theory of Gravity** with corresponding (smoothly invariant) field equations.

Beyond this, in spite of the aspirations of unified field theorists, fields other than gravity/metric need to be considered. Such fields are locally maps from spacetime to some other space, real or complex vectors, spinors, etc. The modern analysis of such structures is expressed in bundle theory.

- **Bundle (Gauge) relativity:** Physical fields and their equations are described locally by maps from open neighborhoods (patches) of spacetime into a value space, called the **fiber**, for each physical field. The bundle or gauge relativity principle states that the physical laws for these fields should be invariant under different local constructions and has been especially fruitful in investigation of quantum force fields. One of these models is the Yang-Mills model which coincidentally turned out to be of important significance for the first discoveries of non-trivial smoothness on topological \mathbb{R}^4 .

For most of the twentieth century it has been assumed by physicists that the choice of which coordinate systems are to be **smooth** is trivially determined by the topological coordinates, since most topological models are based on subsets of topologically Euclidean spaces.

Remark 3 However, one of the main points of this paper is to review the physical implications of the mathematical discovery that the choice of smoothness is not necessarily uniquely determined by the topology, even for relatively simple topological spacetime models such as \mathbb{R}^4 , or the closed cosmology, $\mathbb{R} \times S^3$, etc.

**Comments on diffeomorphism classes:
an analogous geometric model.**

The notion of a diffeomorphism equivalence class of smooth manifolds can be fairly abstract and difficult to grasp. So an analogous construction may be helpful. Consider some simple expressions for the metric on \mathbb{R}^2 ,

$$ds^2 = dx^2 + dy^2, \quad (1)$$

$$ds^2 = dx^2 + x^2 dy^2, \quad (2)$$

$$ds^2 = dx^2 + \sin^2 x dy^2. \quad (3)$$

The first of these is clearly the usual Euclidean geometry with flat, zero, curvature. In one sense, these are three *different* metrics. That is, if we identify the symbols (x, y) in the three equations the expressions are different. But, if we consider these symbols to be “dummy” symbols, and replace $(x, y) \rightarrow (r, \phi)$ in (2), we get

$$ds^2 = dr^2 + r^2 d\phi^2, \quad (4)$$

which we can immediately identify with (1) if we set

$$x = r \cos \phi, \quad y = r \sin \phi.$$

So, apart from the obvious coordinate singularity at the origin, (1) and (2) can be regarded as defining the *same* metric in different coordinate systems. From the viewpoint of a physical model, these two expressions then have the same content, again, neglecting the coordinate singularity. However, one of the fundamental problems that led to the development of modern differential geometry was concerned the development of a tool, the curvature scalar, which would distinguish (3) from the other two metrics. This scalar which is 0 for (1) but $1 \neq 0$ for (3) insures that there exists no coordinate transformation taking (1) into (3). Thus,

Remark 4 Curvature provides the tool to determine that there is no diffeomorphism taking (1) to (3).

Thus, these two metrics describe truly distinct geometries and physics.

The basic problem discussed in this paper is a more subtle one, but very similar nevertheless.

Question 4 Given two smooth homeomorphic manifolds, M_1 , M_2 . Does there exist a diffeomorphism between them? Is there some tool for smoothness analogous to curvature for geometry?

Note that this question is asked about two different manifolds, not about different geometries on a given, fixed, manifold, as discussed in the context of equations (1) through (4) above. Physically the diffeomorphism would be a generalized (global) coordinate transformation, so this question determines whether the topology of the manifold, which is the same for both, uniquely determines the smoothness and thus the physics done on them. Quite surprisingly we find that even simple topologies such as \mathbb{R}^4 and $\mathbb{R} \times S^3$ can carry non-diffeomorphic smoothness structures and thus physically inequivalent physical theories.

Unfortunately, unlike geometries for which curvature can be used to determine whether two apparently different geometries are truly different, or simply isometric but expressed in different coordinates, the diffeomorphism invariants are not easily computed. Most recent invariants include the Seiberg-Witten ones, which can be very difficult to compute. However, exotic smoothness is still in its infancy and there is certainly reason to expect significant progress. We should perhaps again repeat the physical significance of these questions:

Remark 5 Two manifolds which are not diffeomorphic represent physically inequivalent situations. Solutions to the Einstein equations on one are physically distinct from those on another. As far as we know, Einstein solutions have only been explicitly stated on manifolds carrying standard, non-exotic, smooth structures, so a vast repertoire of unexplored Einstein spaces must exist.

Remark 6 The diffeomorphism group defines the natural equivalence class for physics and for differential topology.

Fundamental problem is whether or not such an equivalence class is trivial for a given *topological* X . That is

Question 5 Can a given topological space support truly distinct, non-diffeomorphic differentiable structures????

At this point, let us examine the difference between **different**, and **non-diffeomorphic** structures on a given topological manifold, using the real line as an example.

Thus, take $X = \mathbb{R}^1 = \{\mathbf{p}\}$, each element being a single real number. From this comes the “natural” smoothness structure, \mathcal{D}_1 , generated from one coordinate patch, $U = X$, and

$$\phi_1(\mathbf{p}) = p,$$

that is, the coordinate is simply the numerical value associated with the topological point, \mathbf{p} . Similarly, consider two others, $\mathcal{D}_2, \mathcal{D}_3$, generated also from one patch, with the same domain, $U = X$, but with

$$\phi_2(\mathbf{p}) = 2p,$$

and

$$\phi_3(\mathbf{p}) = p^{1/3}.$$

Clearly, \mathcal{D}_1 is not **different** from \mathcal{D}_2 since the maximal atlases generated by both are the same, in fact,

$$\phi_2 \cdot \phi_1^{-1}(p) = 2p \in C^\infty.$$

Nevertheless they are both **different** from \mathcal{D}_3 , since the coordinates are incompatible in the overlap:

$$\phi_3 \cdot \phi_1^{-1}(p) = p^{1/3} \notin C^\infty.$$

The important point however, is that these **different** structures are in fact **diffeomorphic**, and thus equivalent from the viewpoint both of physics and of the mathematics of differential topology. The diffeomorphism is established with the **homeomorphism**, $f : \mathbf{p} \rightarrow \mathbf{p}^3$, so

$$\phi_3 \cdot f \cdot \phi_1^{-1}(p) = \phi_3(f(\mathbf{p})) = (p^3)^{1/3} = p \in C^\infty.$$

In fact,

Remark 7 Any two differentiable structures on \mathbb{R}^1 are diffeomorphic to each other.

In other words, there is essentially only *one* differentiable structure that can be put on \mathbb{R}^1 both mathematically and physically.

The uniqueness of the smoothness structure on \mathbb{R}^1 is probably not too surprising. In fact it can be generalized to

Theorem(Moise): There is one and only one differentiable structure on any topological manifold of dimension $n < 4$.

The case of higher dimensions cannot be settled so generally. However, using Thom cobordism techniques, the special cases of the topologically trivial \mathbb{R}^n , $n > 4$ can be,

Theorem: There is one and only one differentiable structure on \mathbb{R}^n for $n > 4$, namely the standard one.

The standard cobordism results are not applicable for the $n = 4$ case, so it remained as an open question until the early 80's and the arrival of $\mathbb{R}_{\mathbf{e}}^4$'s.

We now look at some easily understood counter-examples.

Weierstrass functions as exotica

A naive conjecture from elementary calculus is that every function which is continuous over some interval must be at least piecewise smooth, i.e., its derivative exists except at isolated points. "Physical" intuition might well suggest that this conjecture is valid. However, it is not, as demonstrated by the very nice "Weierstrass" functions, such as

$$W(t) = \sum_0^{\infty} a^k \cos(b^k t),$$

where $|a| < 1$. Clearly, this series is absolutely convergent to a continuous function for all t . However, naive term by term differentiation under the summation results in

$$W'(t) \stackrel{??}{=} - \sum_0^{\infty} (ab)^k \sin(b^k t).$$

If $|ab|$ is chosen to be greater than one, the convergence of this series is dubious at best. In fact, it can be shown rigorously that the derivative of $W(t)$ does not exist anywhere over certain intervals.

Some exotic topological products

Another class of non-intuitive results in low dimensions is provided by **Whitehead spaces**. This work is done in the topological category, even before the imposition of smoothness. However, the result of Moise shows that these spaces have unique smoothness anyway. A Whitehead space, W , is an open, contractible three-dimensional topological manifold which has the following exotic properties:

$$W \neq \mathbb{R}^3,$$

but

$$\mathbb{R}^1 \times W = \mathbb{R}^4.$$

In other words, *it is not correct to assume that when an \mathbb{R}^1 is factored in \mathbb{R}^4 the result will necessarily be \mathbb{R}^3 .*

This too is a profoundly counter-intuitive result. The construction of Whitehead spaces can be visualized using an infinite sequence of twisting tori inside each other. The limit of the infinite iteration of this process produces a set whose complement in \mathbb{R}^3 is a Whitehead space. What the implications of this construction are for the smooth case are not now fully understood, but seem to be highly intriguing. In fact, these spaces are used in handlebody constructions of exotic manifolds.

Complex structures as a toy “physical” model

Complex structures are much more “rigid” and thus more easy to treat. Consider the case of establishing a complex structure on \mathbb{R}^2 . The standard one is generated by one neighborhood with

$$(x, y) \rightarrow x + iy \in \mathbb{C}^1.$$

In this case, diffeomorphisms are replaced by **biholomorphisms**. Consider a different complex structure,

$$(x, y) \rightarrow x - iy.$$

This is certainly different, but the homeomorphism, $(x, y) \rightarrow (x, -y)$ is actually a biholomorphism, so these two are complex equivalent. However, it is easy to construct another one which is not biholomorphic to the standard one, thus an **exotic complex structure**. For example, let $(x, y) \rightarrow (g_x, g_y)$ be some homeomorphism of the plane into the open unit disk and define

$$(x, y) \rightarrow g_x + ig_y.$$

This cannot be biholomorphic (equivalent) to the standard complex structure, since there are no bounded non-constant holomorphic functions in the standard structures, but many in the new one. This can be stretched to the physics of vacuum plane electrostatic fields, where the Maxwell equations are equivalent to the condition that the electric field components expressed as $E_x - iE_y$ must be holomorphic functions. Thus, there would be different “physics” resulting from the exotic complex structure.

Milnor spheres

Fortunately there is a class of manageable exotic structures available in the **smooth** category. These were discovered by Milnor (early 60's). The simplest one is an exotic S^7 . This space can be realized naturally as the bundle space of an $SU(2) \approx S^3$ bundle over S^4 which is compactified \mathbb{R}^4 using a construction of Hopf. From the physics viewpoint, a Yang-Mills field with appropriate asymptotic behavior is a cross section of such a principal bundle. Such fields satisfying Yang-Mills field equations are called **instantons** and turn out to be important later in the story of exotica. For now, however, consider the construction of S^7 as the subset of quaternion 2-space, $\{(q_1, q_2) : |q_1|^2 + |q_2|^2 = 1\}$. There is a natural projection of this space into projective quaternion space, $(q_1 : q_2)$. This space, however, turns out to be nothing more than S^4 . The kernel of this map is the set of unit quaternions, $S^3 \approx SU(2)$. Equivalently, S^7 can be defined by two copies of $(\mathbf{H} - 0) \times S^3$, with identification

$$(q, u) \sim (q/|q|^2, qu/|q|)$$

Milnor was able to generalize this to produce a manifold, Σ^7 by means of the identification

$$(q, u) \sim (q/|q|^2, q^j u q^k / |q|)$$

Milnor was then able to show that if $j+k=1$ the space Σ^7 is *topologically* identical (homeomorphic) to S^7 . However, if $(j-k)^2$ is not equal to 1 mod 7, then Σ^7 is exotic, that is not diffeomorphic to standard S^7 .

Summary to this point

1. “Natural” mathematical assumptions based on physical intuitions need not be correct, e.g., Weierstrass functions, Whitehead spaces, Milnor spheres, etc.
2. Physics needs coordinates, “smoothness structures,” in addition to topology, geometry, etc. These cannot be taken for granted as pre-defined.
3. Conclusion: There is more to “global” than merely topology. This additional richness may have physical significance.

The road to exotica

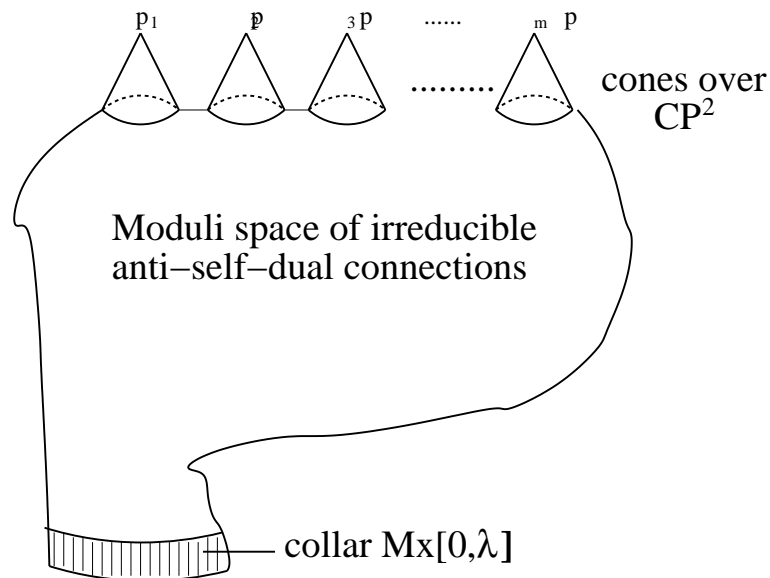
Return to smoothness on \mathbb{R}^4 . The discovery of exotic smoothness on topological \mathbb{R}^4 's, producing manifolds denoted by \mathbb{R}^4_{Θ} , involved developments from many branches of mathematics, including topology and differential equations. Most of these components involve esoteric mathematics. However, one of them is based on the study of **moduli spaces**. This study is based on the physical model of Yang-Mills fields, that is non-Abelian gauge theory. First recall that a moduli space is built from a space of fields, \mathcal{A} , often gauge potentials, over a particular manifold, M . Typically, these fields are further required to satisfy certain field equations and to behave a certain way under gauge transformations, \mathcal{G} . In general \mathcal{A} will be a huge set, certainly not a finite dimensional manifold. So, how can moduli spaces be managed? It turns out that when the gauge transformations are factored out, the result

$$\mathcal{M} = \mathcal{A}/\mathcal{G}$$

can be a well behaved space such as a finite dimensional manifold, perhaps with singularities. \mathcal{M} is a **moduli space**. As a simple example, consider the family of p -forms over a compact manifold, M . Let the field equations be the restriction that the forms be closed. Let the action of the gauge group be the addition of an exact form. The resulting \mathcal{M} in this case is just the p^{th} deRham cohomology group, which is typically a finite dimensional vector space. This is only a simple, not realistic example. More productive is the study of instantons over S^4 , which are certain cross sections of the Hopf bundle, S^7 , as investigated by **Atiyah** and others. These studies lead to:

Remark 8 **The moduli space of certain fields over a manifold can give information about the differential topology of the manifold.**

In the following figure, \mathcal{M} , the moduli space of Yang-Mills connections mod gauge over a compact, simply connected, oriented four manifold, M , is itself a four dimensional manifold, smooth except for isolated singularities at P_1, \dots, P_n , which contains a “collared” copy of M .



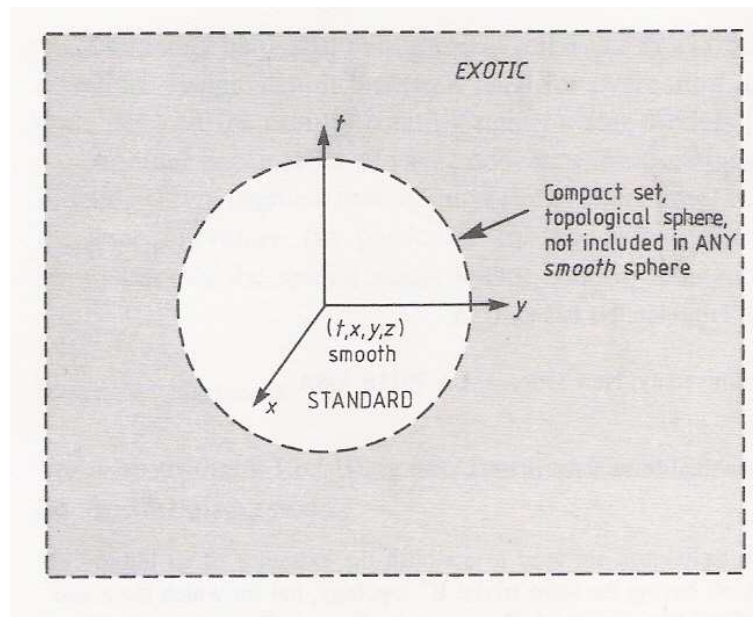
This leads to **Donaldson's theorem**, which restricts the possible topological properties of four manifolds which can also carry a smooth structure.

N.B. This is a amended version(4/16/08) of the talk delivered at the AEI, correcting errors on this page in the previous display of this document. Unfortunately the details of the construction are fairly involved and technical, so it is best to refer to some other source, e.g., the introductory section of the book *Instantons and Four-Manifolds*, Freed and Uhlenbeck, or our book *Exotic Smoothness and Physics*, section §8.4, pp 239 ff.

Suffice it to say here that the construction results in a topological \mathbb{R}^4 , which we denote by \mathbb{R}_{Θ}^4 . By a theorem of Quinn such a non-compact four-manifold has a smooth structure. However, Freedman's construction shows that \mathbb{R}_{Θ}^4 contains a compact subset which cannot be contained in **any** smoothly embedded 3-sphere! Since \mathbb{R}^4 is covered by one smooth coordinate system, $\{x^i, i = 1, \dots, 4\}$, the series of smooth three spheres, $\sum_i (x^i)^2 = R^2$ centered at the origin must eventually contain every compact space for sufficiently large R , this \mathbb{R}_{Θ}^4 must be "exotic," that is, not be diffeomorphic to the standard smooth \mathbb{R}^4 . This surprisingly counter-intuitive result came as a result of the work of many people and came to fruition in papers by Freedman in the early 1980's, which we only briefly review here.

This construction involves the algebraic surface **K3**, a smooth closed topological four manifold, whose topology (specifically, intersection form) is described by the matrices $2E_8 \oplus 3H$. Effectively, the removal 3H part, η_K contained in three copies of $S^2 \times S^2$, and replacement of a neighborhood of η_K by a smooth four-disk results in a manifold with intersection form $2E_8$ which cannot be smoothable since E_8 cannot be diagonalized over the integers. However, consideration of the diffeomorphic image of η_K , that is, $\eta \subset S^2 \times S^2$, results in a topological \mathbb{R}^4 , denoted by \mathbb{R}_{Θ}^4 above, containing a compact set which cannot be itself contained in **any** smoothly embedded S^3 . The reason can be summarized by saying that such an S^3 would have a smooth image in **K3** which would be the boundary of a smooth D^4 , resulting in a smooth manifold with intersection form $2E_8$, contradicting the Donaldson result.

This fact turned out to be of key importance in the road to the discovery of \mathbb{R}_{Θ}^4 . **Donaldson** used moduli space studies to show that spaces with certain intersection forms (a topological feature) could not be smoothed. **Freedman** built on Casson handlebody construction and resulting smooth cobordism in five dimensions. Ultimately the result was the discovery of a topological \mathbb{R}^4 containing a compact set which itself could not be contained in any **smooth** S^3 .



Remark 9 Such a space could thus not be standard, and was the first example of an \mathbb{R}_{Θ}^4 .

Gompf has expanded the early results and produced what he called an “exotic menagerie” of infinitely many non-diffeomorphic \mathbb{R}_{Θ}^4 's, each a topological \mathbb{R}^4 , but with no two diffeomorphic to each other. Gompf's construction makes extensive use of handlebody chains.

So, we ask:

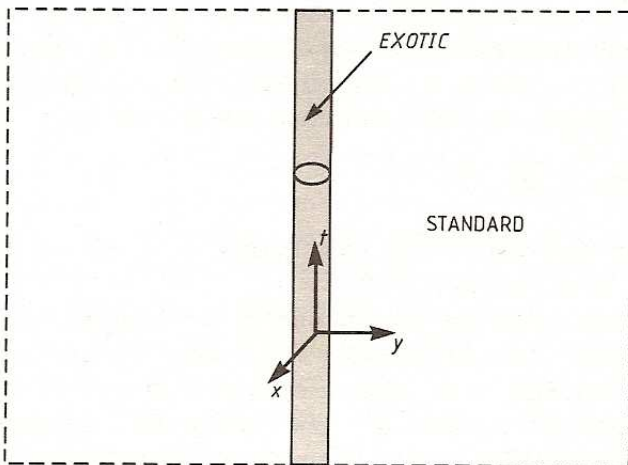
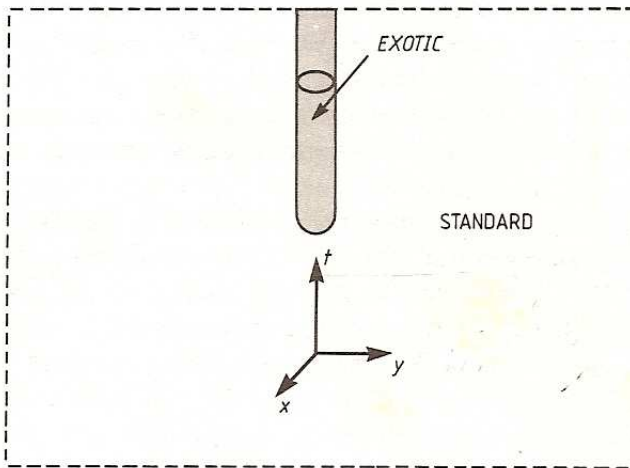
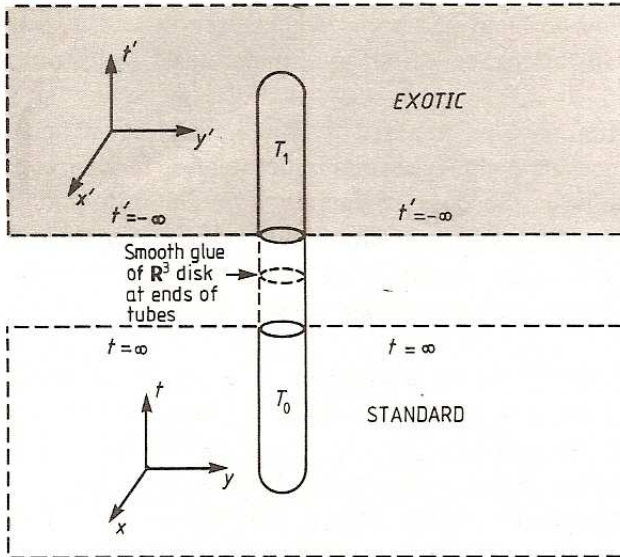
Question 6 What are the differential geometric consequences of exotic smoothness?

The short answer to this is that the consequences are global, and refer to the extendibility of local coordinates and the way in which they are patched together. Recall the early concerns with the Schwarzschild singularity at $r = 2GM$. It was soon discovered that rather than an essential singularity, the anomalous behavior of the metric components at $r = 2GM$ was actually a result of extending the spherical coordinate too far toward the origin. In fact, an alternative extension uses other coordinates, such as proposed by Kruskal et al. The differential geometry depends on the “global” topological question of whether or not $r = 2GM$ is a static sphere, $S^2 \times \mathbb{R}$, if we choose standard Schwarzschild as global (a differential topological choice!), or the set $S^2 \times \{(u, v) | u^2 = v^2\}$, if we choose Kruskal coordinates. Perhaps even more notable is the way in which the $r = 0$ singularity is interpreted. In standard Schwarzschild coordinates it is simply a point times time, $\{pt\} \times \mathbb{R}$, while in Kruskal coordinates it is a hyperboloid, $S^2 \times \{(u, v) | u^2 - v^2 = -1\}$. In other words,

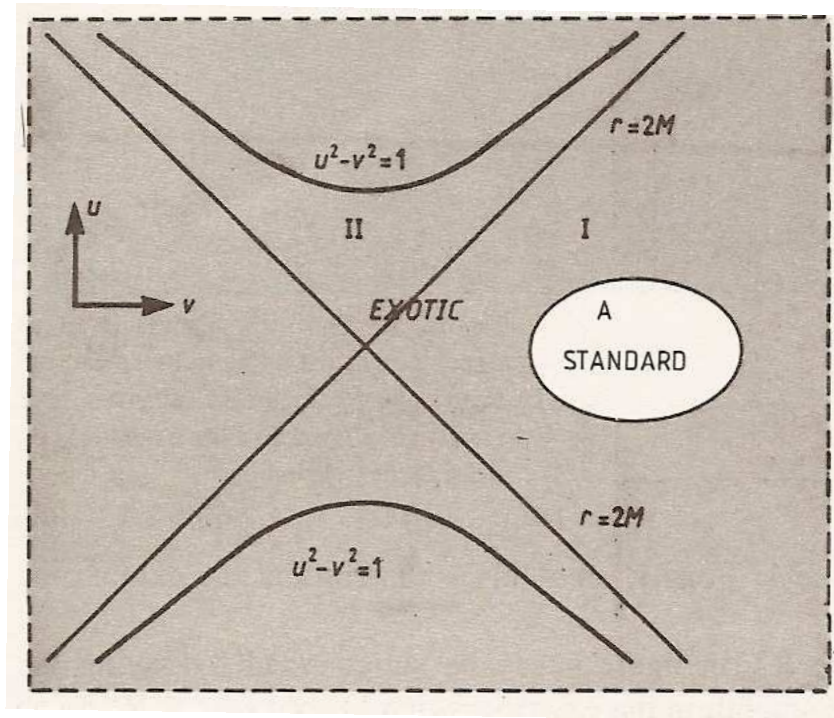
Remark 10 Differential topology, the choice of the way in which local coordinates are patched together, influences the physical properties of the spacetime model supporting differential geometry.

Remark 11 No finite effective coordinate patch presentation exists of any exotic \mathbb{R}^4 .

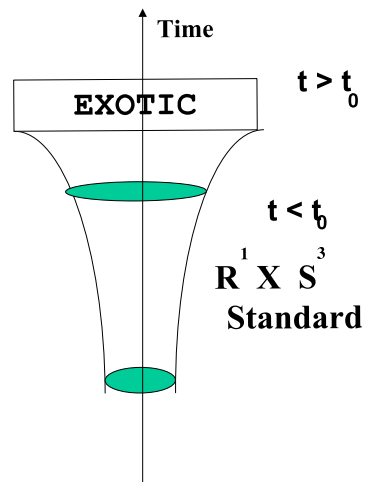
On the physics side, it is possible to use end-sum techniques to produce “*spatially localized exoticness*”.



The physical applications (apart from exotic spheres as models of exotic Yang-Mills discussed earlier) involve \mathbb{R}^4 as spacetime with coordinates (t, x, y, z) etc.



This figure in Kruskal coordinates, shows explicitly that, although standard smoothness may be valid in a macroscopically accessible domain, in some sense the “exoticness” may be localized behind the horizon. This suggests the term “exotic black hole.”



Punctured \mathbb{R}_{Θ}^4 , i.e., $\mathbb{R}^1 \times_{\Theta} S^3$, as a model for exotic cosmology.

Even though there is presently no explicit coordinate patch representation of any \mathbb{R}_{Θ}^4 , there has been

Progress

- Seiberg-Witten gauge theory provides more manageable techniques for probing smoothability and diffeomorphism classes.
- Smoothability of elliptic surfaces. Kirby calculus. Fintushel and Stern relate SW and Donaldson invariants for these.
- Fintushel and Stern use surgery along a knot and link to produce non-diffeomorphic but homeomorphic 4-folds.
- Punctured \mathbb{R}_{Θ}^4 , i.e., $\mathbb{R}^1 \times_{\Theta} S^3$, as a model for **exotic cosmology**.
- Asselmeyer is using tools of Harvey, Lawson and Stingley to relate exoticness to appearance of singularities in connections.

And much more...

Not even wrong?

Of course, given the current controversies surrounding string theories this is a timely question. So we ask:

Question 7 *Are there any physically testable consequences of exotic smoothness?*

The answer to our question is “no” at present. However, this is not because any predictions associated with a non-standard spacetime model would necessarily be too small to be observed as with string theory, but rather that the current mathematical technology has not been able to give us explicit coordinate presentations which would lead to testable predictions. So, this difficulty is not intrinsic to the models, but rather a function of the inadequacy of our mathematical presentation of them. It is reasonable to expect that further study might lead to the tools we need to make physically testable predictions.

Reference/Shameless advertisement:

Check out: *Exotic Smoothness and Physics: Differential Topology and Spacetime Models*, Torsten Asselmeyer-Maluga and Carl H. Brans, World Scientific, 2007.