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Carl H. Brans

New level of relativity

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Abstract In their origins Einstein's studies of relativity principles called into question the validity of important assumptions that had previously been made in formulating physical theories, assumptions made without investigation into alternatives. Examples of this include notions of absolute time and space, flat Euclidean geometry, and trivial topology. In this paper, we review an intermediate niche, differentiable (smooth) structure, which must be defined between topology and geometry. We now know that this choice need not be trivial. Just as it seemed for centuries to be obvious that space should be flat, so it would seem until recently that standard, trivial, smoothness for spacetime is the only choice. We now know that this is not true. In this paper we review these topics in the light of very surprising and often counter-intuitive mathematical discoveries of the last twenty years or so. Since our regions of observability are necessarily constrained we do not have any *a priori* justification for extending standard smoothness globally. This opens up the possibility of non-standard extension of solutions to field equations to exotically smooth regions, leading to examples such as exotic black holes and exotic cosmological models.

Keywords Relativity · Differential Topology

1 Introduction

I am happy and honored to contribute to the volume honoring the work of Bahram Mashhoon. I first met Bahram when we held visiting positions at Jadwin Hall at Princeton in or around 1972. Since then we have gone our

Loyola University
Tel.: +1504-865-3643
Fax:+1504-865-2453
E-mail: brans@loyno.edu

separate ways, but Bahram has continued with in his important work on basic questions related to our understanding of spacetime.

In this paper we review recent mathematical discoveries in differential topology in the context of their implications for basic models of spacetime. We begin with a brief look at the notions of invariance and relativity principles, and their importance over the last century. We will consider the interaction of various relativity principles with assumptions made for the spacetime models, pointing out that there is a niche, choice of differentiability or smoothness, that is not trivial even in topologically trivial Euclidean four space, $\mathbb{R}^4$, or cosmologically significant $\mathbb{R} \times S^3$. We will illustrate the ideas involved in the definition of smoothness with various examples. We will construct a toy model of the physics of two dimensional electrostatics defined in terms of complex structures on the plane where we know there are precisely two inequivalent ones. We will also look at some other examples of surprising, "exotic," structures in topology such as the Whitehead product. We then proceed with a brief overview of the mathematics that led to the discovery of exotic smoothness on topological $S^7, \mathbb{R} \times S^3$, and $\mathbb{R}^4$. We will point out some related physically motivated mathematics (Yang-Mills, gauge theory, etc.) that have been closely involved in these discoveries.

Finally, we close by pointing out that these studies do not of themselves lead to a *new* physical theory, but do motivate us to reconsider fundamental and important *a priori* assumptions made in describing general relativity and other physical field theories. These assumptions have not been available for discussion, since the existence of alternatives was not known until recently. Perhaps the best analogy is the fact that the use of Euclidean geometry without question in describing the behavior of particles and fields over several centuries is now known to be an unjustified assumption after the advent of general relativity. Pursuing this analogy, replace non-Euclidean or non-flat geometry with exotic smoothness. Thus, compared to the previously assumed standard smoothness on spacetime models, solutions in one neighborhood cannot be globally extended to full $\mathbb{R}^4$, or in the cosmological model to the full $\mathbb{R} \times S^3$ without explicit and unjustified assumption. We will briefly consider these issues for exotic black holes and exotic cosmology.

2 Spacetime models, Invariance of theories, and Relativity principles

"Spacetime" serves as the background on which to do physics. See [1], [2], [3]. Early workers, accustomed to a physical ether, wanted it to have substance like qualities, perhaps. So, Einstein's relativity principles, special and general, caused early workers, such as Kretschmann, [2] to consider possibilities of developing unique, invariant ways of identifying points in spacetime through geometric quantities. This later was developed in part by Cartan, then the Petrov classification of directions and scalars, and post-Petrov of Brans[4]. What is notable here is that, in a certain sense, these were searches for an absolute spacetime, even in the midst of the birth of Einstein's relativity principles.

– **Terminology:** Let us note some common terminology used in this paper. **Smooth** and **differentiable** are synonyms meaning differentiable to all orders, sometimes denoted by C^∞. A **diffeomorphism** is a smooth map which is one-to-one and onto with smooth inverse. The term **manifold** can be regarded as a generic term describing a point set which has some standard "local" structure. The most common types are **topological** and **differentiable**. $\mathbb{R}^n$ is the set of points, $\{x^1, ..., x^n\}$, each an ordered set of n real numbers with the standard Euclidean metric topology. A set, M, is a topological manifold if it is a topological space with an open covering of sets, U, together with maps, $\phi_U : U \to \mathbb{R}^n$, which are homeomorphisms onto standard Euclidean space. A topological manifold for which in each non-empty overlap, $U \cap V$, the maps $\phi_U \cdot \phi_V^{-1}$ are diffeomorphisms in terms of the *standard* smooth structure on $\mathbb{R}^n$ defines M as a **smooth** manifold. The word "standard" is emphasized because there was thought to be only one such structure until recently. This is true for all $n \neq 4$, but the non-triviality for $n = 4$ is a major consideration in this paper. Such non-standard smooth manifolds are denoted by $\mathbb{R}^4_{\boldsymbol{\Theta}}$. A **relativity principle** generally is an expression that the laws of physics should be invariant in form, that is, have the same formal structure, under allowed coordinate transformations. Since these laws are generally expressed in terms of differential equations, the permitted coordinate transformations must be expressed at least locally as **diffeomorphisms**. Thus, once we have chosen a possible topology for our spacetime model, we must choose a smoothly consistent family of coordinate systems on it, i.e., a smooth structure. The coordinates provide local tools to express functions as maps from real variables to real, complex, vector, spinor, etc., values. A field theory expressed as differential equations then requires that the field can be differentiated with respect to local coordinates. The relativity principle requires that these expressions can be done in some class (depending on the relativity principle) of permitted coordinates. However, the fact that the various coordinate systems are equivalent means that the field equations can be expressed in any of them, and thus the transformations must be diffeomorphisms.

Remark 1 *The "new level" of relativity referred to in the title then appears here, in the choice of differentiable, or smooth, structure for a given topological spacetime model. Recent results show that in many standard choices for this topological model, the choice of smoothness which determines the class of physically allowable coordinates, is* **not trivial**.

Consider an example. Let us suppose with the early Einstein that our spacetime model is topologically $\mathcal{M}_{PS} = \mathbb{R}^4 = \{p^\mu, \mu = 0, 1, 2, 3\}$ this notation means that $\mathcal{M}_{PS}$ is simply the set of "points" each of which is an ordered set of four real numbers. This is clearly a mathematical model. But, what does it have to do with physics? At this juncture it is easy to get overly involved in philosophy, asking questions about what spacetime "really" is. However, we simply say that this point set is a "model" for "real" spacetime in the sense that we describe our physical theories using this model as a background. In classical point particle mechanics, each point of $\mathcal{M}_{PS}$ is the possible location of an event in the history of a point particle. In more modern field theory, $\mathcal{M}_{PS}$ is the background for expressing field equations. But, again, we leave to the philosophers the task of ascribing some objective reality to the model.

Somewhere in our analysis we explicitly or implicitly associate the notions of "nearness," or "approaching," to some properties of the purely point set, $\mathcal{M}_{PS}$. These physically motivated notions are associated to the mathematical notions defined by a **topology** to be assigned to the point set, $\mathcal{M}_{PS}$ converting it to $\mathcal{M}_{TOP}$. Without much reflection it seems, physicists have chosen the "simplest" topology first axiomatized by mathematicians, that is, the **Euclidean** or real number topology.

In sum, at this juncture we have associated the physically intuitive idea of a point event with an ordered set of four real numbers. Physicists describe the procedure for defining this association as a **coordinate system** or **reference frame** assigning an ordered set of numbers, $\{x^\mu\}$ to each point in $\mathcal{M}_{TOP}$. An obvious choice is simply $x^\mu = p^\mu$. We begin looking into **relativity** principles when we question the uniqueness of this choice, and allow others.

- **Topological relativity:** Here we might assert that our description of physics should be equally valid in any coordinate system that preserves the **topology** of $\mathcal{M}_{TOP}$. Mathematically, this means we must allow any coordinate system which ascribes $\{y^\mu\}$ to points in such a way that the defining functions,
$$\{p^\mu\} \rightarrow \{y^\mu\} = f^\mu(p^\nu)$$
provide a **homeomorphism**, which, by definition, preserves the topology of the model. From Descartes to Newton to the early days of Einstein theory we have assumed that this choice of standard, Euclidean, topology was the appropriate one (and probably the only appropriate one) to be used for spacetime models. Of course, mathematicians have long been interested in spaces with non-euclidean models but it seems it was only when careful consideration was given to certain classes of solutions of the Einstein general relativistic gravitational equations , e.g., 3-sphere closed space cosmology, that alternatives were considered by physicists.

Even at this elementary level, however, we find that we have made a choice of structure for the point set, $\mathbb{R}^4$. That is: **why choose the "standard" topology for $\mathbb{R}^4$?** In fact, there are infinitely many other notions of "nearness" or topology that could be used to define our spacetime model. For example, as a point set, $\mathbb{R}^4$ is isomorphic[1] to $\mathbb{R}^1$, so why not choose this as a basis for the topology of $\mathbb{R}^4$? In other words, why do we choose the product topology of $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ for our spacetime model of events? Why do we choose $\mathcal{M}_{TOP} = \mathbb{R}^4$ for our space time topology, rather than $\mathcal{M}_{TOP} = \mathbb{R}^1$ defined by a Cantor-Peano space filling curve? Is this a result of human intuition, or is there a physical empirical basis? Of course, no viable

[1] This fact was perhaps first made explicit by Peano and relates to set theoretic work of Cantor. See [5]. Another way to define the isomorphism is to interweave the digits representing each of the four numbers, mapping them uniquely into a single number.

theory has used this approach to topologizing the mathematical representation of the four spacetime coordinates, but it is important to realize that an *a priori* choice has been made, and that our topological relativity principle, invariance under homeomorphisms, is restricted to this original choice.

Once the topology has been chosen, we need an additional structure to express point particle kinematics and field theories locally as differential equations. In fact, the physics of Newtonian mechanics was instrumental in the early development of calculus. However, to do calculus we need to know which coordinates are smooth. A choice of which coordinates are to be **smooth** converts $\mathcal{M}_{TOP}$ into $\mathcal{M}_{SMOOTH}$. Given this we are naturally led to another relativity principle:

– **Smooth relativity:** Here the laws of physics must be invariant under smooth transformations of coordinates. That is, if x^μ have been chosen to define smooth coordinates on $\mathcal{M}_{TOP}$ converting it to a $\mathcal{M}_{SMOOTH}$ then the laws of physics must be equally valid in any other coordinate system, y^μ, if the maps defining the y^μ in terms of the x^ν are **diffeomorphisms.** Actually, this can be taken as a definition of Einstein's General Principle of Relativity, but we are getting a little bit ahead of ourselves here because in the meantime flat models of space and spacetime were dominant and thought to be physically appropriate.

Before General Relativity, it seems to have been generally thought that the mathematics of Euclidean geometry for space and Lorentzian geometry for spacetime were the only physically appropriate choices.

– **Geometric relativity:** So, with special relativity Einstein suggested that the laws of physics required the flat spacetime Minkowski metric so that a restricted class of coordinate systems in which this metric assumed its canonical form were physically preferred. These are the inertial reference frames of Special Relativity. The generalization of this to a **Principle of General Relativity** led to arbitrary, but *smooth* coordinate systems. It is of course well understood now that in the context of spacetime this leads to "accelerated" reference frames, freely falling elevators, etc., and eventually to the suggestion that gravity could be associated with a non-trivial metric, **Einstein's General Relativistic Theory of Gravity** with corresponding (smoothly invariant) field equations.

Beyond this, in spite of the aspirations of unified field theorists, fields other than gravity/metric need to be considered. Such fields are locally maps from spacetime to some other space, real or complex vectors, spinors, etc. The modern analysis of such structures is expressed in bundle theory.

– **Bundle (Gauge) relativity:** Physical fields and their equations are described locally by maps from open neighborhoods (patches) of spacetime into a value space, called the **fiber**, for each physical field. The bundle or gauge relativity principle states that the physical laws for these fields should be invariant under different local constructions and has been especially fruitful in investigation of quantum force fields. One of these models is the Yang-Mill model which coincidentally turned out to be of important significance for the first discoveries of non-trivial smoothness on topological $\mathbb{R}^4$.

For most of the twentieth century it has been assumed by physicists that the choice of which coordinate systems are to be **smooth** is trivially determined by the topological coordinates, since most topological models are based on subsets of topologically Euclidean spaces.

Remark 2 *However, one of the main points of this paper is to review the physical implications of the mathematical discovery that the choice of smoothness is not necessarily uniquely determined by the topology, even for relatively simple topological spacetime models such as* $\mathbb{R}^4$*, or the closed cosmology,* $\mathbb{R} \times S^3$*, etc.*

So, we ask:

Question 1 *What are the differential geometric consequences of exotic smoothness?*

The short answer to this is that the consequences are global, and refer to the extendibility of local coordinates and the way in which they are patched together. Recall the early concerns with the Schwarzschild singularity at $r = 2GM$. It was soon discovered that rather than an essential singularity, the anomalous behavior of the metric components at $r = 2GM$ was actually a result of extending the spherical coordinate too far toward the origin. In fact, an alternative extension uses other coordinates, such as proposed by Kruskal et al. The differential geometry depends on the "global" topological question of whether or not $r = 2GM$ is a static sphere, $S^2 \times \mathbb{R}$, if we choose standard Schwarzschild as global (a differential topological choice!), or the set $S^2 \times \{(u,v)|u^2 = v^2\}$, if we choose Kruskal coordinates. Perhaps even more notable is the way in which the $r = 0$ singularity is interpreted. In standard Schwarzschild coordinates it is simply a point times time, $\{pt\} \times \mathbb{R}$, while in Kruskal coordinates it is a hyperboloid, $S^2 \times \{(u,v)|u^2 - v^2 = -1\}$. In other words,

Remark 3 *Differential topology, the choice of the way in which local coordinates are patched together, influences the physical properties of the spacetime model supporting differential geometry.*

3 Diffeomorphism classes

The notion of a diffeomorphism equivalence class of smooth manifolds can be fairly abstract and difficult to grasp. So an analogous construction may be helpful. Consider some simple expressions for the metric on $\mathbb{R}^2$,

$$ds^2 = dx^2 + dy^2, \tag{1}$$

$$ds^2 = dx^2 + x^2 dy^2, \tag{2}$$

$$ds^2 = dx^2 + \sin^2 x dy^2. \tag{3}$$

The first of these is clearly the usual Euclidean geometry with flat, zero, curvature. In one sense, these are three *different* metrics. That is, if we identify the symbols (x, y) in the three equations the expressions are different. But, if we consider these symbols to be "dummy" symbols, and replace $(x, y) \to (r, \phi$ in (2), we get

$$ds^2 = dr^2 + r^2 d\phi^2, \tag{4}$$

which we can immediately identify with (1) if we set

$$x = r\cos\phi, \quad y = r\sin\phi.$$

So, apart from the obvious coordinate singularity at the origin, (1) and (2) can be regarded as defining the *same* metric in different coordinate systems. From the viewpoint of a physical model, these two expressions then have the same content, again, neglecting the coordinate singularity. However, one of the fundamental problems that led to the development of modern differential geometry was concerned the development of a tool, the curvature scalar, which would distinguish (3) from the other two metrics. This scalar which is 0 for (1) but $1 \neq 0$ for (3) insures that there exists no coordinate transformation taking (1) into (3). Thus,

– *There is no diffeomorphism taking (1) to (3).*

Thus, these two metrics describe truly distinct geometries and physics.

The basic problem discussed in this paper is a more subtle one, but very similar nevertheless.

– *Given two smooth homeomorphic manifolds, M_1, M_2. Does there exist a diffeomorphism between them?*

Note that this question is asked about two different manifolds, not about different geometries on a given, fixed, manifold, as discussed in the context of equations (1) through (4) above. Physically the diffeomorphism would be a generalized (global) coordinate transformation, so this question determines whether the topology of the manifold, which is the same for both, uniquely determines the smoothness and thus the physics done on them. Quite surprisingly we find that even simple topologies such as $\mathbb{R}^4$ and $\mathbb{R} \times S^3$ can carry non-diffeomorphic smoothness structures and thus physically inequivalent physical theories.

Unfortunately, unlike geometries for which curvature can be used to determine whether two apparently different geometries are truly different, or simply isometric but expressed in different coordinates, the diffeomorphism invariants are not easily computed. Most recent invariants include the Seiberg-Witten ones, which can be very difficult to compute. However, exotic smoothness is still in its infancy and there is certainly reason to expect significant progress. We should perhaps again repeat the physical significance of these questions:

Remark 4 *Two manifolds which are not diffeomorphic represent physically inequivalent situations. Solutions to the Einstein equations on one are physically distinct from those on another.*

As far as we know, Einstein solutions have only been explicitly stated on manifolds carrying standard, non-exotic, smooth structures, so a vast repertoire of unexplored Einstein spaces must exist.

In summary, consider some of the steps that have been taken to construct modern theories:

– •*spacetime should be an absolute product, time × space,*
– •*spacetime should be geometrically flat,*
– •*spacetime should have trivial topology,*

and perhaps others. Questioning these "natural" assumptions obviously has led to many rich discoveries. In this questioning spirit, it would seem to be well worthwhile to explore the recent discovery of exotic differentiable structures on topologically trivial spaces, especially $\mathbb{R}^4$.

4 Spacetime Structures

To repeat, physical theories need some model of space time, including at least:

1. *Point set*
2. *Topology*
3. *Smoothness, differentiability,* C^∞
4. *Geometry, bundle structures, etc.*

Until now, this transition,

$$point\ set \rightarrow topological\ space \stackrel{??}{\rightarrow} smooth\ manifold \rightarrow bundles, etc.$$

was thought to be fairly well understood, and explored.

However, there has recently been a big surprise, the discovery of *exotic smoothness*. Now it is known that spaces with even very simple topology can support a myriad of inequivalent smoothness structures. This calls into question the triviality of the transition,

$$topological\ space \stackrel{??}{\rightarrow} smooth\ manifold.$$

So, we review the subject of **Differential Topology**, that is, global calculus, directly concerned with this issue.

In defining any point set, X, there may not be *a priori* any *preferred* way to associate numbers with a given point, $\mathbf{p} \in X$. For spacetime models, $\mathbf{p}$ is an **event**. The process of assigning numbers is determined by the physical procedure associated with a **reference frame**, mathematically by a **coordinate patch**.

It is easy to get lazy and falsely secure about this matter since most spaces, X, considered in both physics and mathematics are modeled by subsets of $\mathbb{R}^n$, so each $\mathbf{p}$ is "naturally" associated with an ordered set of real numbers. However, it is well known that

Remark 5 *The choice of coordinates is not unique.*

From this arises the following question at the heart of the **principle of general relativity**:

Question 2 *Does global re-coordination have any physical (or mathematical) consequences?*

The process of assigning numbers to points in X is accomplished by an **atlas of charts**, $(U, \phi_U : U \to \mathbb{R}^n)$. To do calculus, need differential consistency in overlap, $\phi_U \cdot \phi_V^{-1} \in C^\infty$. A **differentiable structure** on X is defined by a maximal atlas, and makes X into a **differentiable**, or **smooth manifold**. The atlas enables the consistent definition of calculus over X, obviously indispensable for any physical theory.

The atlas contains coordinate transitions *within* a given X, but also allows the definition of a natural *equivalence* established by a **diffeomorphism**. This is a homeomorphism of one smooth manifold to another (or itself), $f : X \to Y$, which together with its inverse is smooth when expressed in the atlases on X and Y respectively.

Remark 6 *The diffeomorphism group defines the natural equivalence class for physics and for differential topology.*

The fundamental problem is whether or not such an equivalence class is trivial for a given *topological* X. That is

Question 3 *Can a given topological space support truly distinct, that is, non-diffeomorphic differentiable structures?*

At this point, let us examine the difference between **different**, and **non-diffeomorphic** structures on a given topological manifold, using the real line as an example.

Take $X = \mathbb{R}^1 = \{\mathbf{p}\}$, each element being a single real number. From this comes the "natural" smoothness structure, $\mathcal{D}_1$, generated from one coordinate patch, $U = X$, and

$$\phi_1(\mathbf{p}) = p,$$

that is, the coordinate is simply the numerical value associated with the topological point, $\mathbf{p}$. Similarly, consider two others, $\mathcal{D}_2, \mathcal{D}_3$, generated also from one patch, with the same domain, $U = X$, but with

$$\phi_2(\mathbf{p}) = 2p,$$

and

$$\phi_3(\mathbf{p}) = p^{1/3}.$$

Clearly, $\mathcal{D}_1$ is not **different** from $\mathcal{D}_2$ since the maximal atlases generate by both are the same, in fact,

$$\phi_2 \cdot \phi_1^{-1}(p) = 2p \in C^\infty, \quad \text{a diffeomorphism of } X \text{ onto itself.}$$

Nevertheless they are both **different** from $\mathcal{D}_3$, since the coordinates are incompatible in the overlap:

$$\phi_3 \cdot \phi_1^{-1}(p) = p^{1/3} \notin C^\infty.$$

The important point however, is that these **different** structures are in fact **diffeomorphic**, and thus equivalent from the viewpoint both of physics and of the mathematics of differential topology. The diffeomorphism is established with the **homeomorphism,** $f : \mathbf{p} \to \mathbf{p}^3$, so

$$\phi_3 \cdot f \cdot \phi_1^{-1}(p) = \phi_3(f(\mathbf{p})) = (p^3)^{1/3} = p \in C^\infty.$$

In fact,

Remark 7 *Any two differentiable structures on* $\mathbb{R}^1$ *are diffeomorphic to each other.*

In other words, there is essentially only *one* differentiable structure that can be put on $\mathbb{R}^1$ both mathematically and physically. The uniqueness of the smoothness structure on $\mathbb{R}^1$ is probably not too surprising. In fact it can be generalized to

Theorem(Moise): There is one and only one differentiable structure on any topological manifold of dimension $n < 4$.

The case of higher dimensions cannot be settled so generally. However, using Thom cobordism techniques, the special cases of the topologically trivial $\mathbb{R}^n, n > 4$ can be,

Theorem: There is one and only one differentiable structure on $\mathbb{R}^n$ for $n > 4$, namely the standard one.

The standard cobordism results are not applicable for the $n = 4$ case, so it remained as an open question until the early 1980's and the arrival of $\mathbb{R}^4_{\Theta}$'s.

5 Early topological exotica

We now review some counter-intuitive results from elementary topology to illustrate that assumptions which seem on their face to be obvious may indeed not be valid.

Weierstrass functions

A naive conjecture from elementary calculus is that every function which is continuous over some interval must be at least piecewise smooth, i.e., its derivative exists except at isolated points. "Physical" intuition might well suggest that this conjecture is valid. However, it is not, as demonstrated by the very nice "Weierstrass" functions, such as

$$W(t) = \sum_0^\infty a^k \cos(b^k t),$$

where $|a| < 1$. Clearly, this series is absolutely convergent to a continuous function for all t. However, naive term by term differentiation under the summation results in

$$W'(t) \stackrel{??}{=} -\sum_0^\infty (ab)^k \sin(b^k t).$$

If $|ab|$ is chosen to be greater than one, the convergence of this series is dubious at best. In fact, it can be shown rigorously that the derivative of $W(t)$ does not exist anywhere over certain intervals.

Complex structures as a model

Complex structures are much more "rigid" and thus more easy to treat. Consider the case of establishing a complex structure on $\mathbb{R}^2$. The standard one is generated from one neighborhood by

$$(x, y) \to x + iy \in \mathbb{C}^1.$$

In this case, diffeomorphisms are replaced by **biholomorphisms**. Consider a different complex structure,

$$(x, y) \to x - iy.$$

This is certainly different, but the homeomorphism, $(x, y) \to (x, -y)$ is actually a biholomorphism, so these two are complex equivalent, that is, biholomorphic. However, it is easy to construct another one which is not biholomorphic to the standard one, thus an *exotic* complex structure. For example, let $(x, y) \to (g_x, g_y)$ be some homeomorphism of the plane onto the open unit disk and define

$$(x, y) \to g_x + i g_y \quad \text{with: } g_x^2 + g_y^2 < 1.$$

This cannot be biholomorphic (equivalent) to the standard complex structure, since there are no bounded non-constant holomorphic functions in the standard structures, but many in the new one. This can be extended to the physics of vacuum plane electrostatic fields, where the Maxwell equations are equivalent to the condition that the electric field components expressed as $E_x - iE_y$ must be holomorphic functions. Thus, there would be different "physics" resulting from the exotic complex structure.

Some exotic topological products

Another class of non-intuitive results in low dimensions is provided by **Whitehead spaces**. This work is done in the topological category, even before the imposition of smoothness. A Whitehead space, W, is an open, contractible three-dimensional topological manifold which has the following exotic properties:

$$W \neq \mathbb{R}^3,$$

but

$$\mathbb{R}^1 \times W = \mathbb{R}^4.$$

In other words, *it is not correct to assume that when an* $\mathbb{R}^1$ *is factored in* $\mathbb{R}^4$ *the result will necessarily be* $\mathbb{R}^3$.

This too is a profoundly counter-intuitive result. The construction of Whitehead spaces can be visualized using an infinite sequence of twisting tori inside each other. The limit of the infinite iteration of this process produces a set whose complement in $\mathbb{R}^3$ is a Whitehead space. What the implications of this construction are for the smooth case are not now fully understood, but seem to be highly intriguing. In fact, these spaces are used in handlebody constructions of exotic manifolds.

6 The first exotic smoothness: Milnor spheres

Fortunately there are a class of manageable exotic structures available in the **smooth** category. These were discovered by Milnor in the late 1950's. See [6] for a review by Milnor of his early work in light of recent developments. The simplest one is an exotic S^7. This space can be realized naturally as the bundle space of an $SU(2) \approx S^3$ bundle over S^4, which is compactified $\mathbb{R}^4$, using a construction of Hopf. From the physics viewpoint, a Yang-Mills field with appropriate asymptotic behavior is a cross section of such a principal bundle. Such fields satisfying Yang-Mills field equations are called **instantons** and turn out to be important later in the story of exotica. For now, however, consider the construction of S^7 as the subset of quaternion 2-space, $\{(q_1, q_2) : |q_1|^2 + |q_2|^2 = 1\}$. There is a natural projection of this space into projective quaternion space, $(q_1 : q_2)$. This space, however, turns out to be nothing more than S^4. The kernel of this map is the set of unit quaternions, $S^3 \approx SU(2)$. Equivalently, S^7 can be defined by two copies of $(\mathbf{H} - 0) \times S^3$, with identification

$$(q, u) \sim (q/|q|^2, qu/|q|)$$

Milnor was able to generalize this to produce a manifold, Σ^7 by means of the identification

$$(q, u) \sim (q/|q|^2, q^j u q^k/|q|)$$

Milnor was then able to show that if $j + k = 1$ the space Σ^7 is *topologically* identical (homeomorphic) to S^7. However, if $(j - k)^2$ is not equal to 1 mod 7, then Σ^7 is *exotic*, that is not diffeomorphic to standard S^7.

Summary to this point

- 1) "Natural" mathematical assumptions based on physical intuitions need not be correct, e.g., Weierstrass functions, Whitehead spaces, Milnor spheres, etc.
- 2) Physics needs coordinates, "smoothness structures," in addition to topology, geometry, etc. These cannot be taken for granted as pre-defined.
- 3) Conclusion: There is more to "global" than merely topology. This additional richness may have physical significance.

7 The road to exotic $\mathbb{R}^4_\Theta$

Return to smoothness on $\mathbb{R}^4$. The discovery of exotic smoothness on topological $\mathbb{R}^4$'s, producing manifolds denoted by $\mathbb{R}^4_\Theta$, involved developments from many branches of mathematics, including topology and differential equations. One of the first was based on the study of **moduli spaces**. This study is based on the physical model of Yang-Mills fields, that is non-Abelian gauge theory. First recall that a moduli space is built from a space of fields, $\mathcal{A}$, gauge potentials, over a particular manifold, M. Typically, these fields are further required to satisfy certain field equations and to behave a certain way under gauge transformations, $\mathcal{G}$. In general $\mathcal{A}$ will be a huge set, certainly not a finite dimensional manifold. So, how can moduli spaces be managed? It turns out that when the gauge transformations are factored out, the result

$$\mathcal{M} = \mathcal{A}/\mathcal{G}$$

can be a well behaved space such as a finite dimensional manifold, perhaps with singularities. $\mathcal{M}$ is a **moduli space**. As a simple example, consider the family of p-forms over a compact manifold, M. Let the field equations be the restriction that the forms be closed. Let the action of the gauge group be the addition of an exact form. The resulting $\mathcal{M}$ in this case is just the p^{th} deRham cohomology group, which is typically a finite dimensional vector space. This is only a simple example. More productive is the study of instantons over S^4, which are certain cross sections of the Hopf bundle, S^7, as investigated by **Atiyah** and others.

These studies lead to:

Remark 8 *The moduli space of certain fields over a manifold can give information about the differential topology of the manifold*

This fact turned out to be of key importance in the road to the discovery of $\mathbb{R}^4_{\Theta}$. **Donaldson** used moduli space studies to show that spaces with certain intersection forms (a topological feature) could not be smoothed. **Freedman** built on Casson handlebody construction and resulting smooth cobordism in five dimensions. Ultimately the result was the discovery of a topological $\mathbb{R}^4$ containing a compact set which itself could not be contained in any **smooth** S^3. Such a space could thus not be standard, and was the first example of an $\mathbb{R}^4_{\Theta}$.

Gompf has expanded the early results and produced what he called an "exotic menagerie" of infinitely many non-diffeomorphic $\mathbb{R}^4_{\Theta}$'s, each a topological $\mathbb{R}^4$, but with no two diffeomorphic to each other. Gompf's construction makes extensive use of handlebody chains. However, to date, there remains the sad fact that

Remark 9 *No finite effective coordinate patch presentation exists for any exotic* $\mathbb{R}^4_{\Theta}$.

The original $\mathbb{R}^4_{\Theta}$ were identified because they contained compact set which could not be included in the interior of any smoothly embedded three-sphere. Thus, the exotica was not localized.

The physical applications (apart from exotic spheres as models of exotic Yang-Mills discussed earlier) involve $\mathbb{R}^4$ as spacetime with coordinates (t, x, y, z) etc. For a sketch of this see Figure 1 in standard coordinates.

Figure 2 displays the same idea in Kruskal coordinates, showing explicitly that, although standard smoothness may be valid in a macroscopically accessible domain, in some sense the "exoticness" my be localized behind the horizon. This suggests the term "exotic black hole."

The idea of spatially confining the exoticness can be made explicit by using Gompf's end sum techniques as illustrated in Figure 3.

Nevertheless there has been

Progress

- On the mathematical side,
  - • Seiberg-Witten gauge theory provides more manageable techniques for probing smoothability and diffeomorphism classes.
  - • Smoothability of elliptic surfaces. Kirby calculus. Fintushel and Stern relate SW and Donaldson invariants for these.
  - • Fintushel and Stern use surgery along a knot and link to produce non-diffeomorphic but homeomorphic 4-folds.

  And much more...
- We might mention a few other possible physical applications:

- – • Punctured $\mathbb{R}^4_\Theta$, i.e., $\mathbb{R}^1 \times_\Theta S^3$, as a model for **exotic cosmology.**
- – • Asselmeyer is using tools of Harvey, Lawson and Stingley to relate exoticness to appearance of singularities in connections.

8 Not even wrong?

Of course, given the current controversies surrounding string theories this is a timely question. So we ask:

Question 4 *Are there any* **physically testable** *consequences of exotic smoothness?*

The answer to our question is "no" at present. However, this is not because any predictions associated with a non-standard spacetime model would necessarily be too small to be observed as with string theory, but rather that the current mathematical technology has not been able to give us explicit coordinate presentations which would lead to testable predictions. So, this difficulty is not intrinsic to the models, but rather a function of the inadequacy of our mathematical presentation of them. It is reasonable to expect that further study might lead to the tools we need to make physically testable predictions. Some rough suggestions along these lines have been provided in various studies. For example, [7] looks at the question of extending certain known solutions to the Einstein equations on standard $\mathbb{R}^4$ to a spatially compact region containing exotic smoothness, such as an extension of the Schwarzschild to produce an exotic black hole. In [8] similar questions are looked at in the cosmological context of exotic $\mathbb{R} \times_\Theta S^3$ which is certainly standard for certain neighborhoods of time, i.e., regions of the first factor, $\mathbb{R}$, but will not be for all time if the product is exotic. Asselmeyer [9] has used tools developed by Harvey and Lawson, [10] to investigate the production of mass like singularities from maps of which arise in singular maps from one smooth manifold to another non-diffeomorphic one.

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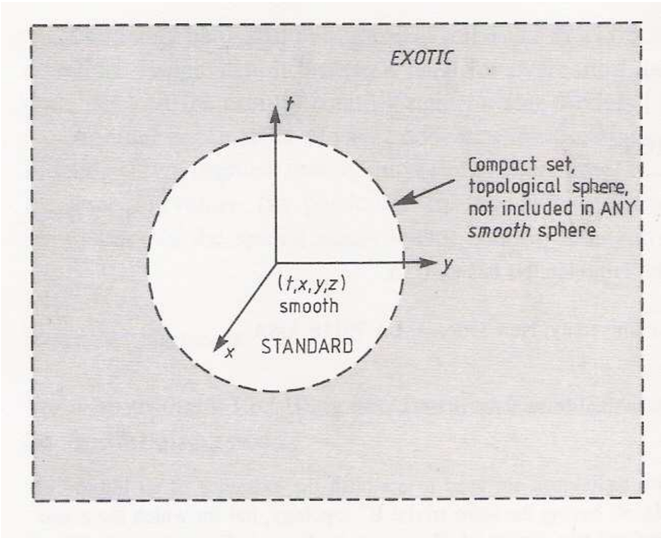

Fig. 1 First examples of $\mathbb{R}^4_\Theta$.

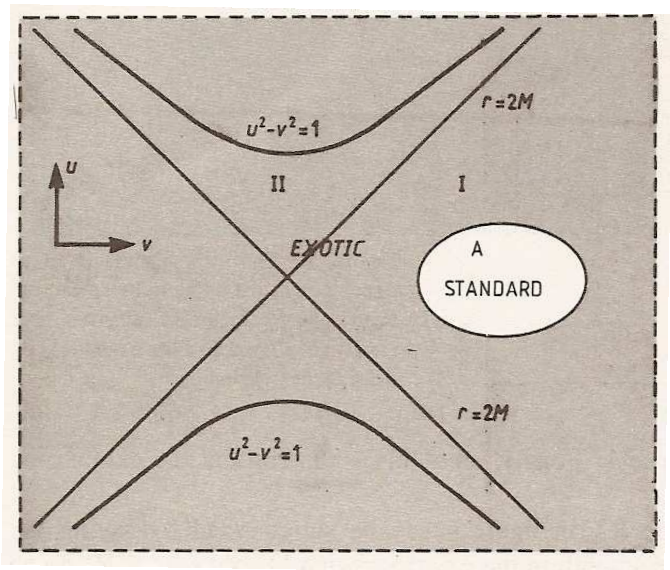

Fig. 2 Localized exotica in Kruskal coordinates, exotic black hole

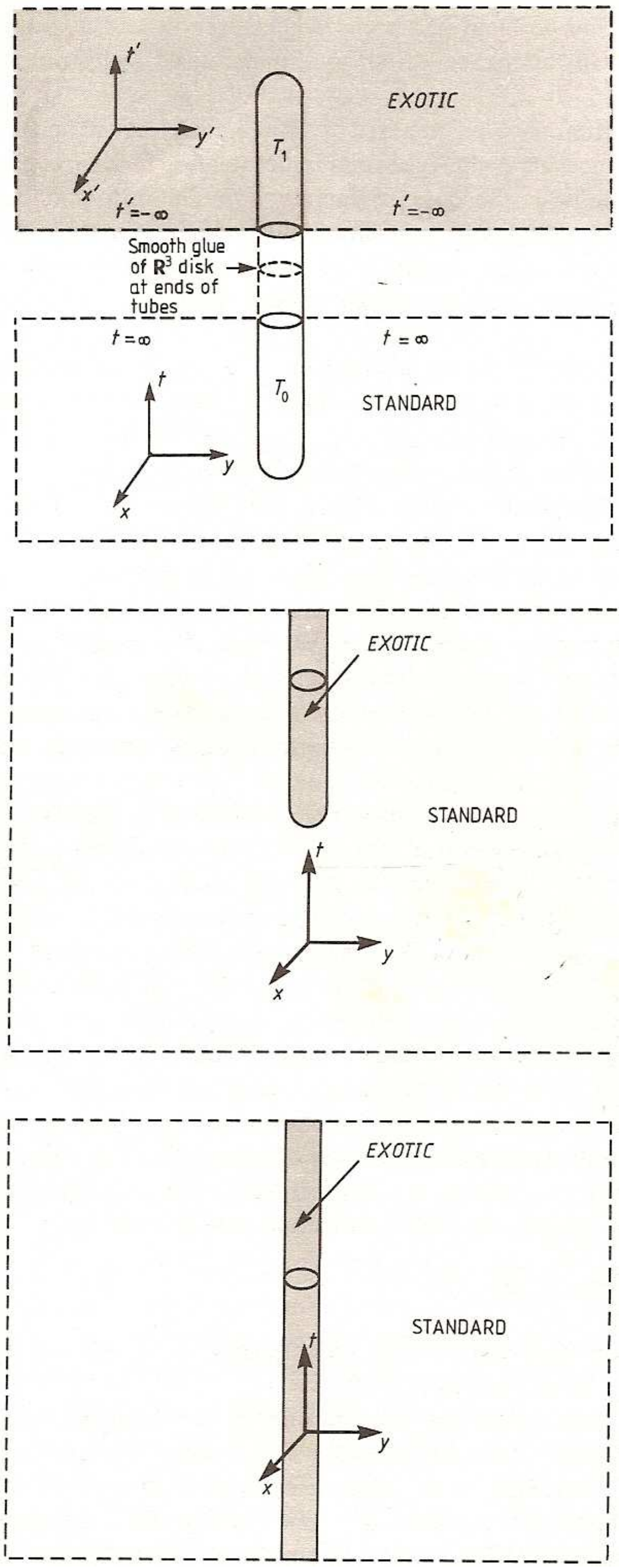

Fig. 3 End sum: three steps to" localize" exotica